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Factorization of the 'classical Boussinesq' system

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Abstract. The scalar equation equivalent to the 'classical Boussinesq', or Broer–Kaup system, is shown to factorize into a differential operator acting on Burgers equation. The linearizability of this latter equation provides a very simple explanation for the recently found property of fusion and fission of a class of solitons for this system.

1. Introduction

The Broer–Kaup (BK) system [1–3], also called the classical Boussinesq system, consists of two coupled partial differential equations (PDE) in the variables (U, V) [1]

$$U_t + (V + \tfrac{1}{2}U^2)_x = 0 \tag{1a}$$

$$V_t + (\beta^2 U_{xx} + UV)_x = 0 \qquad \beta \text{ constant} \tag{1b}$$

or $(U, V - U^2/4 = W)$ [4]

$$U_t + (W + \tfrac{3}{4}U^2)_x = 0 \tag{2a}$$

$$W_t + \beta^2 U_{xxx} + \tfrac{1}{2}UW_x + U_x W = 0 \tag{2b}$$

or $(U, V + \beta U_x = Y)$ [5]

$$U_t + (Y + \tfrac{1}{2}U^2 - \beta U_x)_x = 0 \tag{3a}$$

$$Y_t + (\beta Y_x + UY)_x = 0 \tag{3b}$$

(note that, in equations (1a), (1b) and (2a), (2b), β only appears by its square). It admits the two reductions $V = \pm\beta U_x$ to a Burgers equation for U. Under the natural parametric representation of the first equation

$$U = -\frac{2u_x}{3b_0} \qquad V = \frac{2u_t}{3b_0} - \frac{U^2}{2} \qquad \beta^2 = \frac{a^2}{9b_0^2} \qquad (a, b_0) \text{ constant} \tag{4}$$

the BK system is equivalent to the scalar equation [2]

$$E \equiv \tfrac{1}{3}(a^2 u_{xxx} - 2u_x^3)_x + 2b_0(u_t u_{xx} + 2u_x u_{xt}) - 3b_0^2 u_{tt} = 0. \tag{5}$$

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This equation is invariant by parity $(u, x, t) \to (-u, -x, -t)$ and, in order to avoid writing $\pm a$, we denote by a either of the two square roots of a^2.

This system possesses a Lax pair [2, 6, 7] and admits N-soliton solutions [3] with a coupling factor given in equation (4.16) of [8]. But it also admits another class of solutions of a very particular type: they are expressed analytically as degenerate N-soliton solutions whose coupling factor is zero [9, 10]; these solutions have the remarkable property that they allow fusion and fission to take place [9, 10].

The arguments given up to now for the behaviour of these degenerate solutions are based on the existence of bilinear [10] or even trilinear [9, 11] forms for the BK equation. Despite the correct observation [10] that 'the soliton–fusion solution of BK represents the confluence of shock waves in the Burgers equation [12]', only quite heavy explanations have been given for this phenomenon. One is that 'the BK system is a reduction of the two-component KP hierarchy' [10, 13], another one is that this is just a '(reflection of) the particular structure of the trilinear form' [9].

In this paper, we give a straightforward explanation, which involves no high-level techniques. We show that this phenomenon is an immediate consequence of the factorization of the scalar equation (5) into some operator acting on a *linearizable* equation, namely the Burgers equation.

2. The Burgers subequation

Sachs [14] pointed out the existence of a reduction of the BK system to Burgers equation noticeable on the bilinear representation [8] of the BK system. In the appendix, we show that the Painlevé analysis of (5) provides another reason for the appearance of Burgers equation: the 'singular manifold equation' of (5) is identical to that of Burgers equation (see equation (A10*c*)).

On the scalar form (5) of the BK system, one notices immediately the factorization

$$E \equiv \tfrac{1}{3}\left(-3b_0\partial_t - a\partial_x^2 + 2u_x\partial_x + 2u_{xx}\right) F \qquad F \equiv 3b_0u_t - au_{xx} - u_x^2. \tag{6}$$

This factorization breaks the invariance of equation (5) by parity on (u, x, t), or by parity on a, which is the same.

The linearizability of the Burgers equation into the heat equation [15]

$$u = a \operatorname{Log} \varphi \qquad 3b_0\varphi_t - a\varphi_{xx} = 0 \tag{7}$$

now has the following simple consequences.

Taking for φ the linear superposition

$$\varphi = \sum_{j=1}^{N+1} e^{k_j x + \omega_j t + \delta_j} \qquad 3b_0\omega_j - ak_j^2 = 0 \qquad \delta_j \text{ constant} \tag{8}$$

one generates by equation (7) N-soliton solutions for both Burgers and BK with a zero coupling factor, i.e. of the degenerate type which exhibits fusion and fission.

3. The two sets of solutions to BK

The solutions to BK are thus split into two disjoint subsets:
(i) those which are also solutions of the Burgers subequation;
(ii) those which are not.

The natural language to characterize these two subsets is that of the singularity structure of BK (see the appendix). Indeed, the first subset has one family of movable singularities (i.e. described by one τ function), while the second one has two families (i.e. described by two τ functions).

This explains why, with two τ functions [8, 10] one finds the N-soliton solution to BK, while with one τ function [9, 11] one can only find the degenerate solution with a zero coupling factor. Since the trilinear formalism for the BK system introduces only one τ function, it cannot find the correct N-soliton solution.

Let us illustrate this very important point on the one-soliton solution.

The travelling-wave reduction $u = \int^{x-ct}(Z(\xi) - z_1)\mathrm{d}\xi - (\mu/3b_0)t$, where c and μ are constants and z_1 is a convenient constant translation, yields the equation

$$a^2 Z'^2 = Z^4 - 2(z_1^2 - \mu)Z^2 + K_1 Z + K_2 \qquad \text{for } z_1 = \tfrac{3}{2} b_0 c \tag{9}$$

where K_1, K_2 are two constants of integration. Its general solution Z is single valued. When the four zeros of the right-hand side polynomial are distinct, this is a Jacobi elliptic function; for one double and two simple zeros, the solution is the one-soliton solution [2] of the BK system

$$a^2 Z'^2 = (Z - z_0)^2 \left[(Z + z_0)^2 + 2(z_0^2 - z_1^2 + \mu)\right] \qquad (c, z_0, \mu) \text{ arbitrary} \tag{10}$$

$$Z = z_0 - \frac{3z_0^2 - z_1^2 + \mu}{z_0 \pm \sqrt{\frac{1}{2}\left(3z_1^2 - z_0^2 - 3\mu\right)} \cosh k\xi} \qquad (z_0^2 - 3z_1^2 + 3\mu)(3z_0^2 - z_1^2 + \mu) \neq 0 \tag{11}$$

$$k^2 = \frac{2(3z_0^2 - z_1^2 + \mu)}{a^2} \qquad K_1 = 4z_0(z_1^2 - z_0^2 - \mu) \qquad K_2 = z_0^2(3z_0^2 - 2z_1^2 + 2\mu)$$

and this solution is not a solution of the Burgers equation. For two double zeros, this is a kink solution which is also a solution of the Burgers equation

$$Z = -z_0 \tanh \frac{z_0}{a}\xi \qquad \mu = z_1^2 - z_0^2 \qquad K_1 = 0 \qquad K_2 = z_0^4 \tag{12}$$

and corresponds to $N = 1$ in (8).

When its RHS has two double zeros, the Jacobi equation (9) admits the Riccati subequation

$$aZ' = (Z - z_0)(Z + z_0). \tag{13}$$

The Jacobi equation has two families of movable singularities, and the Riccati equation only one.

The fundamental difference between the true one-soliton (11) and the degenerate solution (12) is that the latter is the logarithmic derivative of the entire function (8) with $N = 1$, while the former is the difference of two logarithmic derivatives of entire functions [16]. Their structure of movable singularities is therefore quite different: a simple pole for (13); two simple poles with opposite residues for (9). There is exactly the same difference between the singularities of u_x depending on whether u satisfies Burgers or BK.

4. Conclusion

The essential feature of the Broer–Kaup system is to have two families of movable singularities, not just one like Burgers. Since, as shown in this paper, the BK system contains the Burgers equation as a subequation, the solutions found to BK depend crucially on the assumption made to find them. The trilinear formalism (assumption of one family) will only find solutions which are also solutions of Burgers, i.e. those which exhibit fusion and fission. The bilinear formalism with two τ functions, or any other assumption with two τ functions [16–20], will find the physically interesting solutions, that is, the ones which are not also solutions of Burgers.

For the same reason, in order to obtain the Lax pair of BK from Painlevé analysis, one must go beyond [18, 19] the SME method, which makes use of only one family. This will be done in a forthcoming paper.

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Appendix. Modified Boussinesq systems

The PDE

$$E \equiv \tfrac{1}{3}(a^2 u_{xxx} - 2u_x^3)_x + 2b_0 u_t u_{xx} - b_1 u_x u_{xt} - 4b_2 u_x u_{xx} + 2b_3 u_{xx} + b_4 u_{xt} + b_5 u_{tt} = 0 \tag{A1}$$

with (a, b_i) constants, is invariant by parity on (u, x, t) (the parameter a represents this invariance). It includes, as particular cases [8], the BK equation (5), the modified Boussinesq equation (MBq) [8, 21] and the modified Korteweg–de Vries (MKdV) equation. Its Painlevé analysis [14, 22] is just a transposition of that of the MBq equation [23, 24]. Let us use the invariant formulation [25] of this analysis, and take as expansion variable a function χ (and the x-primitive $\operatorname{Log}\psi$ of χ^{-1}) whose gradients are

$$\chi_x = 1 + \tfrac{1}{2}S\chi^2 \qquad \chi_t = -C + C_x\chi - \tfrac{1}{2}(CS + C_{xx})\chi^2 \tag{A2}$$

$$(\operatorname{Log}\psi)_x = \chi^{-1} \qquad (\operatorname{Log}\psi)_t = -C\chi^{-1} + \tfrac{1}{2}C_x \tag{A3}$$

$$S_t + C_{xxx} + 2C_x S + CS_x = 0. \tag{A4}$$

The singularity degrees of u and E are 0 and 4:

$$u \sim u_{01}\operatorname{Log}\psi \qquad E \sim 2(u_{01} - a^{-2}u_{01}^3)\chi^{-4} \tag{A5}$$

and the two families $u = a\operatorname{Log}\psi + u_0 + u_1\chi + \cdots$ have the same indices $(-1, 0, 3, 4)$. Each family generates the necessary conditions for the absence of movable logarithms

$$Q_3 \equiv -\frac{1}{4a}(b_1^2 + 4b_5 - 4b_0^2)(C_t + CC_x) = 0 \tag{A6}$$

$$Q_4 \equiv \left[\frac{(b_1-2b_0)(3b_1+2b_0)}{4}b_2+\frac{3b_1b_4}{4}+b_2b_5 + \left(\frac{2b_0-7b_1}{4}-\frac{(b_1-2b_0)^2(3b_1+2b_0)}{16}\right)C\right](C_t+CC_x) + \frac{a}{4}\left[(b_1^2+2b_0b_1-2b_0^2+2b_5)(C_{xt}+CC_{xx})+(b_1^2+4b_5-4b_0^2)C_x^2\right] + (b_5-b_0(b_0+b_1))(u_{0,tt}+C^2u_{0,xx}+(C_t+CC_x)u_{0,x}+2Cu_{0,xt})=0 \qquad \text{(A7)}$$

where u_0 is the arbitrary function introduced at index 0. The other family generates the conditions obtained by changing the sign of a in the above expressions. All these conditions are independent of S, and their resolution for arbitrary (C, u_0) provides the only three solutions:

$$b_0=b_1=b_5=0 \qquad E \equiv \left(\tfrac{1}{3}(a^2u_{xxx}-2u_x^3)-2b_2u_x^2+2b_3u_x+b_4u_t\right)_x=0 \qquad \text{(A8}a\text{)}$$

$$b_0\neq 0 \qquad b_1=0 \qquad b_5=b_0^2 \qquad \text{(A8}b\text{)}$$

$$b_0\neq 0 \qquad b_1=-4b_0 \qquad b_5=-3b_0^2 \qquad b_4=4b_0b_2 \qquad \text{(A8}c\text{)}$$

i.e. respectively the MKdV, MBq and BK PDEs after some linear transformation on u. Since all three have a Lax pair, the necessary conditions are also sufficient.

Let us also determine the 'singular manifold equation' (SME) [26], i.e. the condition on (S, C) which is necessary for the existence of an expansion $u_T = a\,\mathrm{Log}\,\psi + u_0$ restricted to the singular part of one among the two families. This is achieved by eliminating u_0 between the two truncation equations

$$E_1 \equiv 4b_2+(2b_0-b_1)C+4u_{0,x}=0 \qquad \text{(A9}a\text{)}$$

$$E_2 \equiv -2b_3-\frac{a^2}{3}S+b_4C-b_5C^2+a(b_1-b_0)C_x - 2b_0u_{0,t}+4b_2u_{0,x}-b_1Cu_{0,x}+2u_{0,x}^2-2au_{0,xx}=0 \qquad \text{(A9}b\text{)}$$

and results in

$$\text{(MKdV)}: \qquad a^2S-3b_4C+6(b_3+b_2^2)=0 \qquad \text{(A10}a\text{)}$$

$$\text{(MBq)}: \qquad C_t-\left(\frac{C^2}{2}-\frac{b_4}{b_0^2}C-\frac{a^2}{3b_0^2}S\right)_x=0 \qquad \text{(A10}b\text{)}$$

$$\text{(BK)}: \qquad C_t+\left(\frac{C^2}{2}-\frac{2a}{3b_0}C_x-\frac{a^2}{9b_0^2}S\right)_x=0. \qquad \text{(A10}c\text{)}$$

These SMEs are identical to those of three one-family PDEs, respectively the KdV equation for a zero value of the spectral parameter [26], the Boussinesq equation [26], and the Burgers equation [26]. Conversely, given one of the three SMEs (A10), the singular manifold method [26, 27], which only introduces one singular manifold, retrieves the linear system associated with the three one-family PDEs (KdV, Bq, Burgers). In order to retrieve the Lax pair of the two-family PDEs (MKdV, MBq, BK), one must extend [18, 19] the method of Weiss [26].

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